4.6 Rank

Consider again the matrix $A_{4\times 5}$ studied in Section 4.3

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 5 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}$$

We showed that A is row-equivalent to $\tilde{A}$ Given by

$$A \sim \tilde{A} = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We define the row vectors of A and $\tilde{A}$ as follows:

$\vec{r}_1 = (1, 2, 3, -4, 8)$ $\quad$ $\vec{q}_1 = (1, 2, 0, 2, 5)$

$\vec{r}_2 = (1, 2, 0, 2, 5)$ $\quad$ $\vec{q}_2 = (0, 0, 3, -6, 3)$

$\vec{r}_3 = (2, 4, -3, 10, 9)$ $\quad$ $\vec{q}_3 = (0, 0, 0, 0, 2)$

$\vec{r}_4 = (3, 6, 0, 6, 9)$ $\quad$ $\vec{q}_4 = (0, 0, 0, 0, 0)$

Notice that

$$
\begin{array}{c}
\vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \\ \vec{r}_4
\end{array}
\begin{bmatrix}
1 & 2 & 3 & -4 & 8 \\
1 & 2 & 0 & 2 & 5 \\
2 & 4 & -3 & 10 & 9 \\
3 & 6 & 0 & 6 & 9
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 3 & -4 & 8 \\
0 & 0 & -3 & 6 & -3 \\
0 & 0 & -9 & 18 & -7 \\
0 & 0 & -9 & 18 & 15
\end{bmatrix}
\begin{array}{l}
\\ \vec{r}_2 - \vec{r}_1 \\ \vec{r}_3 - 2\vec{r}_1 \\ \vec{r}_4 - 3\vec{r}_1
\end{array}
$$

$$
\sim
\begin{bmatrix}
1 & 2 & 3 & -4 & 8 \\
0 & 0 & -3 & 6 & -3 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 22
\end{bmatrix}
\begin{array}{l}
\vec{r}_1 \\ \vec{r}_2 - \vec{r}_1 \\ (\vec{r}_3 - 2\vec{r}_1) - 3(\vec{r}_2 - \vec{r}_1) \\ (\vec{r}_4 - 3\vec{r}_1) - (\vec{r}_3 - 2\vec{r}_1)
\end{array}
\sim
$$

$$
\tilde{A} =
\begin{array}{c}
\begin{array}{ccccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{array} \\
\begin{bmatrix}
1 & 2 & 0 & 2 & 5 \\
0 & 0 & 3 & -6 & 3 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\begin{array}{l}
\\
\overbrace{\vec{r}_1 + (\vec{r}_2 - \vec{r}_1)}^{=\vec{r}_2} = \vec{q}_1 \\
\vec{r}_2 - \vec{r}_1 = \vec{q}_2 \\
\vec{r}_3 - 2\vec{r}_1 - 3(\vec{r}_2 - \vec{r}_1) = \vec{q}_3 \\
\vec{r}_4 - 3\vec{r}_1 - (\vec{r}_3 - 2\vec{r}_1) - 11(\vec{r}_3 - 2\vec{r}_1 - 3(\vec{r}_2 - \vec{r}_1) = \vec{q}_4
\end{array}
$$

What relationship exists between the row vectors of A and $\tilde{A}$?

they are linear combination of each others

<u>Definition</u>

For a given matrix $A_{m\times n}$, the row space of A also called Row (A) is the subspace spanned by all the row vectors of A.

Notice that $Row(A) \subset \mathbb{R}^n$

and $Col(A^T) = Row(A)$.

From previous example, we notice

$$Row(\tilde{A}) \subset Row(A) \qquad (3.1)$$

Similarly, it can be shown that rows of A can be obtained from the rows of $\tilde{A}$ by reversing the row operations performed to obtain $\tilde{A}$ from A. Therefore,

$$Row(A) \subset Row(\tilde{A}) \qquad (3.2)$$

These two conditions (3.1) and (3.2) implies

$$Row(A) = Row(\tilde{A}).$$

This procedure can be applied to any matrix $A_{m\times n}$

<u>Thm 13</u> If $A \sim B$, then their row spaces are the same, or

$$Row(A) = Row(B).$$

Notice that the nonzero rows of $\tilde{A}$ are linearly independent

$\vec{a}_1 = (1, 2, 0, 2, 5)$
$\vec{a}_2 = (0, 0, 3, -6, 3)$
$\vec{a}_3 = (0, 0, 0, 0, 2)$

$\vec{a}_3$ is not lin. comb. of $\vec{a}_2$ and $\vec{a}_1$ is not lin comb. of $\vec{a}_2$ and $\vec{a}_3$.

This is also a general result for any $A_{m \times n}$ matrix and an echelon form $\tilde{A}$.

Then, we have the following result:

<u>Corollary.-</u> The nonzero rows of $\tilde{A}$ form a basis for $Row(A)$.

From $\tilde{A}$, we can easily obtain Nul(A).

In fact,

$$2x_5 = 0 \Rightarrow x_5 = 0$$

$$3x_3 - 6x_4 + 3\cancel{x_5}^{\,0} = 0 \Rightarrow x_3 = 2x_4.$$

$$x_1 + 2x_2 + 2x_4 + 5\cancel{x_5}^{\,0} = 0 \Rightarrow x_1 = -2x_2 - 2x_4$$

or

$$\vec{x} \in Nul(A) \Leftrightarrow \vec{x} = \begin{bmatrix} -2x_2 - 2x_4 \\ x_2 \\ 2x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

So $\dim Nul(A) = 2$

<u>Basis for Row(A), Col(A) and Nul(A)</u>

$$B_{RA} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ -6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^5 \qquad B_{CA} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 9 \\ 9 \end{bmatrix} \right\} \subset \mathbb{R}^4$$

$$B_{NA} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^5$$

Corollary.- $\dim \text{Row}(A) = \dim \text{Col}(A)$, $A_{m\times n}$

Proof.-

Although $\text{Row}(A) \subset \mathbb{R}^n$

and $\text{Col}(A) \subset \mathbb{R}^m$

are completely different subspaces their dimensions are the same.

In fact,

$\dim \text{Col}(A) =$ Number of pivot columns in A

$=$ Number of pivot positions in an echelon form $\tilde{A}$ of A $=$ Number of nonzero rows of $\tilde{A}$.

$= \dim \text{Row}(A)$.

Def.- The rank of A ($\text{rank } A$) is the common dimension of $\text{Col } A$ and $\text{Row } A$.

Remark: The $\dim \text{Nul}(A)$ equals the number of free variables in the equation $A\vec{x} = \vec{0}$.

Equivalently,

$\dim \text{Nul}(A) =$ number of nonpivot columns.

Rank THEOREM

Thm 14. For $A_{m\times n}$

$$\text{rank}(A) + \dim \text{Nul}(A) = n.$$

Proof.-

$$\underset{\substack{\| \\ \dim \text{Col}(A)}}{\begin{matrix}\text{number of} \\ \text{pivot columns} \\ \text{of } A\end{matrix}} + \underset{\substack{\| \\ \dim \text{Nul}(A)}}{\begin{matrix}\text{number of} \\ \text{nonpivot columns} \\ \text{of } A\end{matrix}} = \begin{matrix}\text{Total} \\ \text{number of} \\ \text{columns of } A\end{matrix} = n.$$

$$\dim \text{Col}(A) + \dim \text{Nul}(A) = n.$$

(free vars. in $A\vec{x} = \vec{0}$)

Example 3 (book)

a) If $A_{7\times 9}$ has a two-dimensional null space, what is the rank(A)?

$$\text{rank}(A) = n - \dim \text{Nul}(A) = 9 - 2 = 7.$$

what can you say about solns. for $A\vec{x} = \vec{b}$, for any $\vec{b}$?

b) Could a 6×9 matrix A have a two-dimensional null space?

Using the rank theorem

$$\text{rank}(A) = 9 - 2 = 7$$

but this is impossible because $\text{Col}(A) \subset \mathbb{R}^6$

Thus, $\text{rank}(A) \leq 6$.

Invertible matrix thm

$A_{n\times n}$ invertible if and only if

m) Columns of A form a basis for $\mathbb{R}^n$.

n) $\text{Col}(A) = \mathbb{R}^n$

o) $\dim(\text{Col}(A)) = n$.

p) $\text{rank } A = n$

q) $\text{Nul}(A) = \{\vec{0}\}$.

r) $\dim(\text{Nul}(A)) = 0$.

Proof.

g) $A\vec{x} = \vec{b}$ has at least one soln. for each $\vec{b} \in \mathbb{R}^n$.

$$(g) \Rightarrow (n) \Rightarrow (o) \Rightarrow (p) \overset{\text{Thm 14}}{\Rightarrow} (r) \Rightarrow (q) \Rightarrow (d)$$

$A\vec{x} = \vec{0}$ has only trivial soln.